

Uncertainty related to position and momentum localization of a quantum state

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Abstract This paper presents the uncertainty related to position and momentum localization of a quantum state in terms of entropic uncertainty relations. We slightly improve the inequality given in [Phys. Rev. A 74, 052101 (2006)] and introduce a new entropic measure with corresponding uncertainty relation.

1 Introduction

From the famous Heisenberg uncertainty principle we can learn that „it is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized” [1]. It means that a measurement of two observables dealing with position and momentum localization of a quantum state respectively shall expose some uncertainty. Our goal is to describe this uncertainty (originating from complementarity of position and momentum variables) with the help of the entropic uncertainty relations. To this end we shall use the Rényi entropy defined for a set of probabilities P_i :

$$H_\alpha^{(P)} = \frac{1}{1-\alpha} \ln \left[\sum_i P_i^\alpha \right], \quad (1)$$

where $\alpha > 0$. In some cases we will also use the Shannon entropy defined by the following limit:

$$S^{(P)} = \lim_{\alpha \rightarrow 1} H_\alpha^{(P)} = - \sum_i P_i \ln P_i. \quad (2)$$

To assure the maximal simplicity of the formulas appearing in this paper we will restrict ourselves to a one-dimensional case, where the quantum state is described in the position representation by a normalized wave function $\psi(x)$. In the momentum

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representation the same state is described by the wave function $\tilde{\psi}(p)$ related to the previous one by the Fourier transformation:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dx e^{-ipx/\hbar} \psi(x). \quad (3)$$

To explain how to describe the uncertainty in terms of entropic uncertainty relations let us consider two arbitrary observables (F and G) with corresponding hermitian operators \hat{F} and \hat{G} . We assume that these operators have point spectrum and in general do not commute with each other. The eigenstates $\varphi_m(x)$ of the operator \hat{F} , and $\theta_n(x)$ of the operator \hat{G} form two orthonormal bases:

$$\int_{\mathbb{R}} dx \varphi_m(x) \varphi_{m'}^*(x) = \delta_{mm'}, \quad \int_{\mathbb{R}} dx \theta_n(x) \theta_{n'}^*(x) = \delta_{nn'}. \quad (4)$$

The following completeness relations are also satisfied:

$$\sum_m \varphi_m(x) \varphi_m^*(x') = \sum_n \theta_n(x) \theta_n^*(x') = \delta(x - x'). \quad (5)$$

With the observables F and G we can associate the probability distributions $|f_m|^2$ and $|g_n|^2$, where:

$$f_m = \int_{\mathbb{R}} dx \psi(x) \varphi_m^*(x), \quad (6)$$

$$g_n = \int_{\mathbb{R}} dx \psi(x) \theta_n^*(x). \quad (7)$$

The f_m and g_n vectors are connected by a unitary transformation U_{mn} :

$$f_m = \sum_n U_{mn} g_n, \quad g_n = \sum_m U_{mn}^* f_m, \quad \sum_n U_{mn} U_{m'n}^* = \delta_{mm'}, \quad (8)$$

where:

$$U_{mn} = \int_{\mathbb{R}} dx \varphi_m^*(x) \theta_n(x). \quad (9)$$

According to Riesz theorem [2, 3] we have the norm inequality:

$$\left[c_U \sum_m |f_m|^{2\alpha} \right]^{1/\alpha} \leq \left[c_U \sum_n |g_n|^{2\beta} \right]^{1/\beta}, \quad (10)$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad \alpha \geq 1, \quad c_U = \sup_{(m,n)} |U_{mn}|. \quad (11)$$

By definition $0 \leq c_U \leq 1$. Taking the logarithm of both sides of the norm inequality (10) we obtain the uncertainty relation for the sum of the Rényi entropies:

$$H_{\alpha}^{(F)} + H_{\beta}^{(G)} \geq -2 \ln c_U. \quad (12)$$

For every given pair of observables (F, G) one can find the eigenstates φ_m and θ_n and calculate the value of c_U . Typically, when the commutator of \hat{F} and \hat{G} does not vanish, c_U shall be less than 1, what simply means that the values of the observables F and G cannot be simultaneously well determined. When this commutator is equal to 0 the operators \hat{F} and \hat{G} share common eigenvectors and $c_U = 1$.

2 Localized measurements

To include information about localization properties of the quantum state we shall change our previous set of observables. Let us divide both position and momentum space into equal bins. To describe this partition we introduce the characteristic function:

$$\chi_j(s) = \begin{cases} 1 & s \in [(j - \frac{1}{2})\delta s, (j + \frac{1}{2})\delta s] \\ 0 & \text{elsewhere} \end{cases}. \quad (13)$$

The index j labels the bins and δs denotes the bin's width. With the k th bin in the position space we associate an observable A_k with corresponding eigenstates $\varphi_{km}(x)$ forming in the line segment, described by $\chi_k(x)$, the orthonormal basis:

$$\int_{\mathbb{R}} dx \chi_k(x) \varphi_{km}(x) \varphi_{km'}^*(x) = \delta_{mm'}, \quad \sum_m \varphi_{km}(x) \varphi_{km}^*(x') = \delta_k(x - x'). \quad (14)$$

One should notice that the Dirac delta function in (5) differs from the one in (14) by its domain. The domain of $\delta_k(x - x')$ is restricted to the k th bin. Similarly in the l th bin in the momentum space, described by $\chi_l(p)$, we introduce an observable B_l with eigenstates $\theta_{ln}(p)$. We have:

$$\int_{\mathbb{R}} dp \chi_l(p) \theta_{ln}(p) \theta_{ln'}^*(p) = \delta_{nn'}, \quad \sum_n \theta_{ln}(p) \theta_{ln}^*(p') = \delta_l(p - p'). \quad (15)$$

For the observables A_k and B_l we have the probability distributions $|a_{km}|^2$ and $|b_{ln}|^2$, where:

$$a_{km} = \int_{\mathbb{R}} dx \chi_k(x) \psi(x) \varphi_{km}^*(x), \quad (16)$$

$$b_{ln} = \int_{\mathbb{R}} dp \chi_l(p) \tilde{\psi}(p) \theta_{ln}^*(p). \quad (17)$$

In this case the unitary transformation U_{kmln} :

$$a_{km} = \sum_{l,n} U_{kmln} b_{ln}, \quad b_{ln} = \sum_{k,m} U_{kmln}^* a_{km}, \quad (18)$$

reads:

$$U_{kmln} = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dx \chi_k(x) \int_{\mathbb{R}} dp \chi_l(p) e^{ipx/\hbar} \varphi_{km}^*(x) \theta_{ln}(p). \quad (19)$$

Using the same arguments as before we obtain the uncertainty relation:

$$H_\alpha^{(A)} + H_\beta^{(B)} \geq -2 \ln c_U, \quad c_U = \sup_{(k,m,l,n)} |U_{kmln}|, \quad (20)$$

where now the Rényi entropies are:

$$H_\alpha^{(A)} = \frac{1}{1-\alpha} \ln \left[\sum_{k,m=-\infty}^{\infty} |a_{km}|^{2\alpha} \right], \quad H_\beta^{(B)} = \frac{1}{1-\beta} \ln \left[\sum_{l,n=-\infty}^{\infty} |b_{ln}|^{2\beta} \right]. \quad (21)$$

The uncertainty relation (20) carries both information about the uncertainty of position and momentum localization, and uncertainties associated with the observables A_k and B_l . To calculate the uncertainty related only to the localization properties of the state we shall find the maximal value C_{max} among c_U 's calculated for whole possible choices of $\varphi_{km}(x)$ and $\theta_{ln}(p)$. To do this we will firstly use the Hölder inequality [4]:

$$|U_{kmln}| \leq \frac{1}{\sqrt{2\pi\hbar}} \left(\int_{\mathbb{R}} dx \chi_k(x) |\varphi_{km}(x)|^2 \right)^{1/2} \times \left(\int_{\mathbb{R}} dx \chi_k(x) \left| \int_{\mathbb{R}} dp \chi_l(p) e^{ipx/\hbar} \theta_{ln}(p) \right|^2 \right)^{1/2}. \quad (22)$$

From the orthonormality condition (14) we obtain that the integral in the first parenthesis is equal to 1. Now we rewrite (22):

$$|U_{kmln}| \leq \sqrt{W[\theta_{ln}]}, \quad (23)$$

where we have introduced the functional $W[\theta_{ln}]$ of the form:

$$W[\theta_{ln}] = \int_{\mathbb{R}} dp \chi_l(p) \int_{\mathbb{R}} dq \chi_l(q) Q_k(p-q) \theta_{ln}(p) \theta_{ln}^*(q), \quad (24)$$

and the integral kernel $Q_k(p-q)$ is:

$$Q_k(p-q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} dx \chi_k(x) e^{i(p-q)x/\hbar} = e^{-i\frac{k\delta x}{\hbar}(p-q)} \frac{\sin\left[\frac{\delta x}{2\hbar}(p-q)\right]}{\pi(p-q)}. \quad (25)$$

We can define a new function:

$$\Psi(p) = e^{-i\frac{k\delta x}{\hbar}(p+l\delta p)} \theta_{ln}(p+l\delta p), \quad (26)$$

in terms of which the functional W takes a simpler form:

$$W[\Psi] = \int_{\mathbb{R}} dp \chi_0(p) \int_{\mathbb{R}} dq \chi_0(q) Q_0(p-q) \Psi(p) \Psi^*(q). \quad (27)$$

We have got rid of the indices k and l what could be done due to the translational symmetries both in coordinate and momentum spaces. To find the maximal possible value of the functional $W[\Psi]$ we have to solve the variational equation:

$$\frac{\delta}{\delta \Psi} \left(W[\Psi] - \lambda \int_{\mathbb{R}} dp \chi_0(p) |\Psi(p)|^2 \right) = 0, \quad (28)$$

where λ is the Lagrange multiplier associated with the normalization constraint. The equation (28) leads to the Fredholm integral equation of the second kind [5]:

$$\frac{1}{\pi} \int_{-1}^1 ds \frac{\sin \left[\frac{\gamma}{4}(t-s) \right]}{t-s} \Psi_j(s) = \lambda_j \Psi_j(t), \quad (29)$$

where $\gamma = \delta x \delta p / \hbar$ is a dimensionless parameter which appears instead of the bin's widths δx and δp . The maximal value of $W[\Psi]$ is the largest eigenvalue λ_0 of (29) [1, 5]:

$$W[\Psi] \leq \lambda_0 = \frac{\gamma}{2\pi} \left[R_{00} \left(\frac{\gamma}{4}, 1 \right) \right]^2, \quad (30)$$

where $R_{00}(s, t)$ is one of the radial prolate spheroidal wave functions of the first kind¹ [6]. Thus, with the help of the variational method we have found that:

$$C_{max} = \sqrt{\lambda_0} = \sqrt{\frac{\gamma}{2\pi}} R_{00} \left(\frac{\gamma}{4}, 1 \right). \quad (31)$$

For every finite value of the parameter γ we have $C_{max} < 1$ which is the measure of uncertainty of position and momentum localization. The corresponding entropic uncertainty relation is the Maassen-Uffink type inequality [3, 7]:

$$H_{\alpha}^{(A)} + H_{\beta}^{(B)} \geq -2 \ln C_{max}. \quad (32)$$

3 Probability distributions for localization in coordinate and momentum space

The probability distributions $|a_{km}|^2$ and $|b_{ln}|^2$ carry both information about localization and information about the observables A_k and B_l . In order to obtain a pure information about localization we shall trace out the observables' degrees of freedom:

$$q_k = \sum_m |a_{km}|^2 = \int_{\mathbb{R}} dx \chi_k(x) |\Psi(x)|^2, \quad (33)$$

$$p_l = \sum_n |b_{ln}|^2 = \int_{\mathbb{R}} dp \chi_l(p) |\tilde{\Psi}(p)|^2. \quad (34)$$

¹ in the Wolfram Mathematica's notation it is `SpheroidalS1[0, 0, s, t]`.

The q_k coefficient has an interpretation of the probability of finding the particle in the k 'th bin in the coordinate space. The p_l coefficient can be interpreted in a similar manner. The probability distributions q_k and p_l are affected by the Heisenberg uncertainty principle, thus we have [1]:

$$\forall_{k,l} \quad q_k + p_l \leq 1 + \sqrt{\lambda_0}, \quad (35)$$

where λ_0 was defined in (30). The way of finding the relation (35) leads to the same integral equation (29). From (35) we can easily find that:

$$\forall_{k,l} \quad q_k p_l \leq \frac{1}{4} \left(1 + \sqrt{\lambda_0}\right)^2. \quad (36)$$

The inequality (36) is suitable to be used together with the Shannon entropies, because the sum of the Shannon entropies is:

$$S^{(q)} + S^{(p)} = - \sum_{k,l} q_k p_l \ln q_k p_l, \quad (37)$$

and we are able to find a simple lower bound:

$$S^{(q)} + S^{(p)} \geq - \sum_{k,l} q_k p_l \ln \left[\frac{1}{4} \left(1 + \sqrt{\lambda_0}\right)^2 \right] \geq -2 \ln \left[\frac{1}{2} (1 + C_{max}) \right]. \quad (38)$$

We can call it the Deutsch type inequality [7, 8].

In spite of the fact that $H_\alpha^{(A)} \geq H_\alpha^{(q)}$ and $H_\beta^{(B)} \geq H_\beta^{(p)}$ there is another lower bound for the sum $H_\alpha^{(q)} + H_\beta^{(p)}$ which in the quantum regime ($\gamma < 1$) is significantly better than (32). The probability distributions (33, 34) fulfill the chain of inequalities:

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty} q_k^\alpha \right)^{1/\alpha} &\leq (\delta x)^{1-1/\alpha} \left(\int_{\mathbb{R}} dx |\psi(x)|^{2\alpha} \right)^{1/\alpha} \\ &\leq (\delta x)^{(1-1/\alpha)} \left(\frac{\alpha}{\pi} \right)^{-1/2\alpha} \left(\frac{\beta}{\pi} \right)^{1/2\beta} \left(\int_{\mathbb{R}} dp |\tilde{\psi}(p)|^{2\beta} \right)^{1/\beta} \\ &\leq (\delta x)^{(1-1/\alpha)} (\delta p)^{(1/\beta-1)} \left(\frac{\alpha}{\pi} \right)^{-1/2\alpha} \left(\frac{\beta}{\pi} \right)^{1/2\beta} \left(\sum_{l=-\infty}^{\infty} p_l^\beta \right)^{1/\beta}, \end{aligned} \quad (39)$$

where the first and the last one are the Jensen inequalities and in the middle there is the famous Beckner inequality [9]. From (39) one can derive the uncertainty relation [10]:

$$H_\alpha^{(q)} + H_\beta^{(p)} \geq -\frac{1}{2} \left(\frac{\ln \alpha}{1-\alpha} + \frac{\ln \beta}{1-\beta} \right) + \ln \pi - \ln \gamma. \quad (40)$$

This lower bound is better than (32) and (38) for small values of γ parameter but unfortunately it becomes negative (and in fact meaningless) for:

$$\gamma \geq \frac{\pi}{\alpha} (2\alpha - 1)^{\frac{2\alpha-1}{2(\alpha-1)}}. \quad (41)$$

In the case of the Shannon entropies ($\alpha \rightarrow 1$) the condition (41) reads $\gamma \geq e\pi$.

4 Summary

The best calculated lower bounds for the sum of Shannon entropies which can be treated as a measure of uncertainty of position and momentum localization of the quantum state are:

$$S^{(A)} + S^{(B)} \geq -\ln \left[\min \left\{ \frac{\gamma}{e\pi}, \frac{\gamma}{2\pi} \left[R_{00} \left(\frac{\gamma}{4}, 1 \right) \right]^2 \right\} \right], \quad (42)$$

in the case of the probability distributions $|a_{km}|^2$ and $|b_{ln}|^2$, and:

$$S^{(q)} + S^{(p)} \geq -\ln \left[\min \left\{ \frac{\gamma}{e\pi}, \frac{1}{4} \left[1 + \sqrt{\frac{\gamma}{2\pi}} R_{00} \left(\frac{\gamma}{4}, 1 \right) \right]^2 \right\} \right], \quad (43)$$

in the case of the probability distributions (33, 34) which completely describe the localization properties.

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